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Translation Planes Admitting $SL(n, w)$ with $n \geqslant 3$

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1. INTRODUCTION

In 1974 Lorimer [11], using the isomorphism between A_8 and $GL(4, 2)$, gave the first example of a translation plane having a collineation group $G = SL(n, w)$ for some integer $n \geqslant 3$ and some prime power $w \geqslant 2$. This plane was independently discovered by Rahilly [14] while investigating generalized Hall planes. For the Lorimer–Rahilly plane the order is 16, $n = 3$, and $w = 2$. Walker [15] in 1976 gave a different interpretation of Lorimer's construction; he showed further that Lorimer's construction gives rise to a second translation plane of order 16 admitting $SL(3, 2)$ as a collineation group. This second plane was also discovered by N. L. Johnson in his work on derived semifield planes of order 16. No other translation planes of this type are presently known.

In [7–9] Jha and Kallaher have considered translation planes admitting $G = SL(n, w)$, where $n \geqslant 3$ and w is a prime power, as a collineation group. Under the additional assumption that G acts on the line ℓ_∞ as a tangentially transitive group would, they show that the only possibilities are the Lorimer–Rahilly plane and the Johnson–Walker plane with $n = 3$ and $w = 2$.

A *Lorimer plane* is defined as a translation plane having a collineation group $G = SL(n, w)$, where $n \geqslant 3$ and w is a prime power, contained in its translation complement. The positive integers n and w are called *Lorimer invariants* of the plane. It follows easily that the group G must be contained in the linear translation complement; that is, the group G consists of linear transformations on the plane considered as a vector space over its kernel. In Section 4 Lorimer planes without any additional restriction are investigated. The main result is that the dimension d of a Lorimer plane over its kernel is large relative to n. (Note that as the vector space over its kernel a translation plane has dimension $2d$; the dimension of a line through the origin—the zero

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vector—has dimension d over the kernel. See Section 2D of [10].) Specifically, it is shown that $2^{n-2} | d$ for odd characteristic and $n \leqslant 2d$ for even characteristic. In most cases these inequalities can be improved. See Theorem 4.1 and Corollary 4.2.1.

In particular, the dimension d of a Lorimer plane is always at least 4. (This was proved in [7].) Section 5 looks at Lorimer planes of dimension 4 over their kernel. The main result in this section is: If the plane has characteristic p, then $p > 2$ implies $n = 3$ and $w = 2$, while $p = 2$ implies either $n = w = 3$ or $n \leqslant 8$ and $w = 2^s$ for some $s \geqslant 1$.

These results give support to the following conjecture: if a translation plane of dimension d over its kernel $K = GF(q)$ has a collineation group $G = SL(n, w)$ in its linear translation complement, then both w and q are powers of 2. (This would imply that $SL(n, w)$, with w odd and $n \geqslant 3$, does *not* act on any translation plane.) Furthermore, it is also probably true that $n \leqslant d$. Lorimer, in a private communication, suggested the study of translation planes of dimension 4 over their kernel $K = GF(q)$ possessing $G = SL(n, q)$, where $n \geqslant 3$, as a collineation group. In Section 5 it is shown that for such translation planes $q = 2^k$ for some positive integer k and $n = 3$. In particular, the only translation planes of dimension 4 and kernel $K = GF(p)$, where p is a prime, possessing a collineation group $G = SL(n, p)$ are the Lorimer–Rahilly planes and the Johnson–Walker planes. (Here $n = 3$ and $p = 2$.)

It is assumed that the reader is familiar with the theory of projective planes as exhibited in Dembowski [1] and the theory of translation planes as given in Lüneburg [12]. Familiarity with the background information and nomenclature given in [10] is also assumed; notation and basic concepts explained in [10] will be used freely.

2. Commuting Baer Involutions

In [13] Ostrom proved the following results.

THEOREM 2.1. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime, and let S be an elementary abelian group of order 2^a contained in the linear translation complement $LC(\pi)$. If the non-identity elements of S are Baer involutions, then 2^a divides D.*

Three generalizations of this result will be given in this section. For the first generalization, Theorem 2.2, two lemmas are needed.

LEMMA 2.1. *Let π be a translation plane of dimension d over its kernel*

$K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. Let S be an abelian 2-group contained in the linear translation complement $LC(\pi)$. If S' is the subset of S consisting of involutory homologies, then $|S'| = 0$, 1, or 3 and $S_1 = S' \cup \{1\}$ is a subgroup of S having order 1, 2, or 4.

Proof. Assume S contains four distinct involutory homologies. By Theorem 3.1 of Lüneburg [12] at least three must be affine homologies (that is, have affine axis). Let β_1, β_2, β_3 be three such affine homologies and for each i, let β_i have axis the affine line m_i and center P_i. Note that $P_i \in \ell_\infty$ and $\mathcal{O} = (0, 0) \in m_i$ for each i since $\beta_i \in LC(\pi)$ for each i. If $i \neq j$, then $\beta_i\beta_j = \beta_j\beta_i$ implies β_j fixes the axis m_i and center P_i for β_i. By Corollary 3.4 of Lüneburg [12] the axis m_j of β_j cannot be m_1; hence m_j goes through P_i (that is, $m_j = \mathcal{O}P_i$) and m_i goes through P_j (that is, $m_i = \mathcal{O}P_j$). It follows that $m_2 = m_3 = \mathcal{O}P_1$ and $P_2 = P_3 = m_1 \cap \ell_\infty$. But this again contradicts Corollary 3.4 of Lüneburg [12]. This contradiction shows that $0 \leqslant |S'| \leqslant 3$. A simple argument using Lemma 4.8 of Lüneburg [12] shows that $S' \cup \{1\}$ is a subgroup having order 1, 2, or 4. This proves the lemma.

If the group S of Lemma 2.1 is elementary abelian, then more can be said as the second lemma shows.

LEMMA 2.2. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. Let S be an elementary abelian 2-group contained in the linear translation complement $LC(\pi)$. If S has order 2^c with $c \geqslant 2$, then S has a subgroup S_0 of order 2^{c-2} whose non-identity elements are Baer involutions.*

Proof. Since S is elementary abelian, there exists a subgroup S_0 of S with $S = S_0 S_1$ and $S_0 \cap S_1 = 1$, where S_1 is the subgroup of the conclusion of Lemma 2.1. (Use the fact that S is a vector space over the field Z_2.) Also, the order of S_0 is 2^{c-e}, where $|S_1| = 2^e$. Since $0 \leqslant e \leqslant 2$, the lemma follows.

Our first generalization is a direct consequence of Theorem 2.1 and Lemma 2.2.

THEOREM 2.2. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. If the linear translation complement $LC(\pi)$ contains an elementary abelian 2-group order 2^c with $c \geqslant 2$, then 2^{c-2} divides d.*

For the second generalization of Theorem 2.1 two lemmas are needed. The first one is as follows.

LEMMA 2.3. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. Let S be an* *elementary abelian 2-group contained in the linear translation complement $LC(\pi)$. If a non-identity element τ of S has at least three distinct conjugates (within $LC(\pi)$) in S, then τ is a Baer involution.*

Proof. By Lemma 2.1 the group S can contain at most three distinct involutory homologies. If τ is an involutory homology, then S has three distinct involutory homologies: the involution τ and two non-trivial conjugates of τ. Because S is abelian, the hypothesis of Lemma 4.8 in Lüneburg [12] holds. (See the proof of Lemma 2.1.) Hence, one of the conjugates of τ is the unique involutory homology σ with axis ℓ_∞ and center $\mathcal{O}$. But this is a contradiction since the only conjugate of σ within $LC(\pi)$ is itself. Thus, τ is not an involutory homology; hence, τ is a Baer involution.

The next lemma is an immediate corollary to Lemma 2.3.

LEMMA 2.4. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. Let S be an elementary abelian 2-group contained in the linear translation complement $LC(\pi)$. If every non-identity element of S has at least three distinct conjugates (within $LC(\pi)$) in S, then the non-identity elements of S are Baer involutions.*

Our second generalization follows directly from Theorem 2.1 and Lemma 2.4.

THEOREM 2.3. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime, and let S be an elementary abelian 2-group of order 2^c in the linear translation complement $LC(\pi)$. If every non-identity element of S has at least three distinct conjugates (within $LC(\pi)$) in S, then 2^c divides d.*

A careful perusal of the proof of Theorem 2.2 shows that if $|S| = 2^c$ and $|S_1| = 2^e$, where this notation is that of Lemma 2.1, then 2^{c-e} divides d. This observation is used in the proof of the last result of this section.

THEOREM 2.4. *Let π be a translation plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. Let Σ be a set of commuting involutions contained in the linear translation complement $LC(\pi)$ of π. If either* (1) *Σ consists of Baer involutions or* (2) *the elements of Σ are all conjugate (within $LC(\pi)$) and $s \equiv |\Sigma| \geqslant 3$, then $s + 1 \leqslant 4d$.*

Proof. By Lemma 2.3, condition (2) implies condition (1); hence Σ contains only Baer involutions in both cases. Since $1 \notin \Sigma$, the elementary abelian group S generated by Σ has order at least $s + 1$. By Theorem 2.2 the integer $2^{-e}s$, where $2^e - 1$ is the number of involution homologies in S, divides d. Since $0 \leqslant e \leqslant 2$, the theorem follow.

It is not difficult to construct examples in many different types of translation planes—for example, nearfield planes or Hall planes—which show that Theorems 2.2 and 2.4 are the best possible.

3. 2–Subgroups of $SL(n, w)$

In this section the groups $SL(n, w)$ for $n \geqslant 3$ are considered. The purpose is the determination of some elementary abelian 2-subgroups in $SL(n, w)$. The results of this section, together with those of the previous section, are used in Section 4 to limit the possible dimension of translation planes upon which $SL(n, w)$ can act. The first three theorems consider the case where the prime power w is odd, and the fourth theorem covers the case where w is even.

In this section the natural representation of $SL(n, w)$ as $n \times n$ matrices over the field $GF(w)$ will be used.

THEOREM 3.1. *Let $n \geqslant 3$, and let w be an odd prime power. The group $G = SL(n, w)$ contains an elementary abelian 2-group S of order 2^{n-1} whose elements are diagonal matrices with an even number of -1's on the diagonal and 1's for the other diagonal entries.*

Proof. It is clear that the set S of n by n diagonal matrices over $GF(w)$ with an even number of -1's on the diagonal and all remaining diagonal entries 1's forms an elementary abelian 2-subgroup of $SL(n, w)$. The number of such matrices with -1 occurring $2m$ times on the diagonal is given by the binomial coefficient $\binom{n}{2m}$. Hence,

$$|S| = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2t} = \sum_{m=0}^{t} \binom{n}{2m},$$

where $t = [n/2]$, the greatest integer in $n/2$. Since this sum is just the number of subsets of an n-set with an even number of elements, it follows that

$$|S| = \tfrac{1}{2}(2^n) = 2^{n-1}.$$

The second theorem discusses the number of conjugates an element of the subgroup S given in Theorem 3.1 has in S.

THEOREM 3.2. *Let $n \geqslant 3$, let w be an odd prime power, and let S be the* *elementary abelian 2-group of $SL(n, w)$ described in Theorem* 3.1. *For each $m = 0, 1, \ldots, t = \lfloor n/2 \rfloor$, the matrices in S with -1 occurring $2m$ times on the diagonal are all conjugate within $SL(n, w)$.*

Proof. Note that $m = 0$ gives only the identity matrix I_n which is conjugate only to itself. If $n = 3$, then $t = \lfloor 3/2 \rfloor = 1$ and the non-identity elements of S are conjugate in $SL(3, w)$ by Lemma 4.3 of Jha and Kallaher [7].

Assume now that $n \geqslant 4$. Consider conjugacy in $GL(n, w)$ by permutation matrices. Conjugating an element A of S by a permutation matrix P, in effect, just rearranges the diagonal elements of A. That is, conjugation by P is equivalent to permuting the diagonal elements of A by a permutation ρ in S_n, the symmetric group. Furthermore, the matrix P has determinant 1, that is, $P \in SL(n, w)$, if and only if ρ is in A_n, the alternating group. Let Per be the group of all permutation matrices in $GL(n, w)$ and let $\mathrm{EP} \equiv \mathrm{Per} \cap SL(n, w)$. Then

$$|\mathrm{EP}| = \tfrac{1}{2}(n!). \tag{3.1}$$

Also, the set $\mathrm{OP} \equiv \mathrm{Per} - \mathrm{EP}$ consists of the permutation matrices in $GL(n, w)$ having determinant -1 corresponding to the odd permutations in S_n. For obvious reasons, the elements of EP will be called even, and those of OP will be called odd.

The matrix I_n in S is clearly by itself in a conjugacy class under EP. If n is even, the matrix $-I_n = \mathrm{diag}[-1, -1, \ldots, -1]$ is also by itself in a conjugacy class under EP. Let $1 \leqslant m \leqslant \lfloor n/2 \rfloor$. Consider the matrix

$$B_m = \begin{bmatrix} -I_{2m} & 0 \\ 0 & I_{n-2m} \end{bmatrix}. \tag{3.2}$$

Consider the subgroup of EP consisting of those permutation matrices P with $P^{-1} B_m P = B_m$. Then P must have the form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \tag{3.3}$$

where P_1 is a permutation matrix of size $2m$ and P_2 is an $(n-2m) \times (n-2m)$ permutation matrix. Furthermore, either $\det P_1 = \det P_2 = 1$ or $\det P_1 = \det P_2 = -1$. In the first case P_1 and P_2 must be even permutation matrices, while in the second case they must both be odd. Hence, the number of such matrices P is

$$\tfrac{1}{4}[(2m)!][(n-2m)!] + \tfrac{1}{4}[(2m)!][(n-2m)!] = \tfrac{1}{2}[(2m)!][(n-2m)!]. \tag{3.4}$$

From (3.1) and (3.4) the number of distinct conjugates of B_m under the group EP is

$$\frac{[\frac{1}{2}(n!)]}{\frac{1}{2}[(2m)!][(n-2m)!]} = \binom{n}{2m}. \tag{3.5}$$

Since this is the number of matrices in S with -1 occurring $2m$ times on the diagonal, the theorem is proven.

The third theorem follows directly from Theorem 3.2.

THEOREM 3.3. *Let $n \geqslant 3$, let w be an odd prime power, and let S be the elementary abelian 2-subgroup of $SL(n, w)$ described in Theorem* 3.1. *The following statements hold:*

(i) *If n is odd then every non-identity element of S has at least three conjugates (within $SL(n, w)$) in S.*

(ii) *If n is even then every non-identity element of S, except for $-I_n = \operatorname{diag}[-1, -1, \ldots, -1]$, has at least three conjugates (within $SL(n, w)$) in S. The element $-I_n$ is self-conjugate.*

Proof. Assume n is odd. Let m be an integer with $1 \leqslant m \leqslant t = \lfloor n/2 \rfloor$. Consider the matrix B_m given in (3.2). The number of conjugates of B_m in S is $\binom{n}{2m} \geqslant n \geqslant 3$ by Theorem 3.2. This proves statement (i). Statement (ii) has a similar proof.

The situation with the prime power w even is covered by the fourth theorem of this section.

THEOREM 3.4. *Let $n \geqslant 3$ and let $w = 2^s$ for some integer $s \geqslant 1$. The group $SL(n, w)$ contains an elementary abelian 2-group S of order $(2^s)^{n-1}$ whose non-identity elements are conjugate (within $SL(n, w)$) to each other.*

Proof. Consider the group S of G consisting of all elements of the form

$$\alpha(x_2, \ldots, x_n) = \begin{bmatrix} 1 & x_2 & \cdots & x_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_2 \cdots x_n \\ 0 & \\ \vdots & I_{n-1} \\ 0 & \end{bmatrix}.$$

It is straightforward to show that S is elementary abelian of order $(2^s)^{n-1}$. The non-trivial elements $(x_2, \ldots, x_n)$ are transvections with respect to the hyperplane $H = \{(0, a_2, \ldots, a_n) \mid a_i \in GF(w)\}$ of the vector space $GF(w)^n$. By Hilfssatz 6.9 of Huppert [5] the non-trivial elements of S are conjugate within $SL(n, w)$.

4. Lorimer Planes

This section discusses a class of translation planes called Lorimer planes. They are defined as follows.

Definition 4.1. A *Lorimer plane* is a translation plane π of finite order having a collineation group $G = SL(n, w)$, where $n \geqslant 3$ and w is a prime power, contained in the linear translation complement $LC(\pi)$. The integers n and w are called *Lorimer invariants* of π.

In the definition of Lorimer plane, the group $SL(n, w)$ is assumed to be contained in $LC(\pi)$. It is not a difficult exercise to show that if the translation plane π of characteristic p has the group $SL(n, w)$ acting on it as a collineation group and p does not divide $(n, w-1)$, then $LC(\pi)$ contains a subgroup isomorphic to $SL(n, w)$. An interesting question, which we do not investigate, is the possibility of a Lorimer plane having two sets of Lorimer invariants n_1, w_1 and n_2, w_2 with $(w_1, w_2) = 1$.

As mentioned in the Introduction the only known examples of Lorimer planes are the Lorimer–Rahilly and the Johnson–Walker translation planes of order 16 and having Lorimer invariants $n = 3$, $w = 2$. In both planes the dimension is 4 over the kernel $K = GF(2)$. For both planes, the group $G = SL(3, 2)$ fixes pointwise a set of three points on the line ℓ_∞ and is transitive on the remaining 14 points. In [7–9] Jha and Kallaher investigated Lorimer planes in which the group $G = SL(n, w)$ acted on ℓ_∞ in this manner; they showed that only the Lorimer–Rahilly and Johnson–Walker planes occur. Here no restriction is placed upon the group $G = SL(n, w)$.

The first five results give restrictions on the dimension of a Lorimer plane. The last result restricts how the p-elements in $G = SL(n, w)$, where p is the characteristic of the plane, can act. The results given in Sections 2 and 3 will be used.

Note that when the group $SL(n, w)$ acts on a translation plane of dimension d over its kernel $K = GF(q)$ all matrix representations will consist of $2d \times 2d$ matrixes over K; thus, the natural representation of Section 3 will not be used.

Theorem 4.1. *Let π be a Lorimer plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p and odd prime, and let n and w be Lorimer invariants of π. The following statements hold:*

(i) *If $w = u^s$, where u is an odd prime and n is odd, then 2^{n-1} divides d.*

(ii) *If $w = u^s$, where u is an odd prime and n is even, then 2^{n-2} divides d.*

(iii) *If* $w = 2^s$, *then* $(2^s)^{n-1}$ *divides* d.

In particular, the dimension d *is divisible by* 4.

Proof. Recall that $n \geqslant 3$. Assume first that w is odd. If n is odd then Lemma 2.4 and statement (i) of Theorem 3.1 are Baer involutions. By Theorem 2.1 the order 2^{n-1} of S divides d. If n is even, the statement (ii) of Theorem 3.3 applies. Hence, the group S of Theorem 3.1 has at most one involutory homology. By Theorem 2.2 and the remark after Theorem 2.3 the power 2^{n-2} divides d.

Statement (iii) follows from Theorem 3.4, Lemma 2.4, and Theorem 2.1. The fact that 4 divides d always follows from the fact that $n \geqslant 3$.

The last statement of Theorem 4.1 can be improved if the Lorimer invariant w is odd. To show this the following lemma is needed.

LEMMA 4.1. *Let* π *be a translation plane of dimension* d *over its kernel* $K = GF(q)$, *where* $q = p^k$ *with* $k \geqslant 1$ *and* p *a prime. Let* π_0 *be the Baer subplane fixed pointwise by a Baer involution* α *in the linear translation complement* $LC(\pi)$. *The following statements hold:*

(i) *The subplane* π_0 *is a translation plane with kernel* K_o *containing* K *as a subfield.*

(ii) *If* π_0 *has dimension* d_0 *over its kernel* K_0, *then*

$$d = 2d_0 e,$$

where e *is the dimension of* K_0 *as a vector space over* K.

Proof. By Lemma 2.2 of Jha and Kallaher [7] the plane π_0 is a subspace of π having dimension d over K. Let ℓ be a line through the origin of π such that $\ell_0 = \ell \cap \pi_0$ is a line through the origin $\mathcal{O}$ of π_0. Now ℓ has dimension d as a vector space over K and thus ℓ_0 has dimension $\frac{1}{2}d$ over K. (See Theorem 2.1 of [7] and Section 2d of [10].) Furthermore, the line ℓ_0 is a vector space over K_0 of dimension d_0. Since K is a subfield of K_0, the component ℓ_0 must then have dimension $d_0 e$ as a vector space over K. Hence $\frac{1}{2}d = d_0 e$. This proves the lemma.

THEOREM 4.2. *Let* π *be a translation plane of dimension* d *over its kernel* $K = GF(1)$, *where* $q = p^k$ *with* $k \geqslant 1$ *and* p *an odd prime. If* $G = SL(3, w)$, *where* w *is an odd prime power, is a subgroup of the linear translation complement* $LC(\pi)$, *then* 8 *divides the dimension* d.

Proof. The subgroup of G consisting of the identity and the matrices $\tau_1 = \operatorname{diag}[1, -1, -1]$, $\tau_2 = \operatorname{diag}[-1, 1, -1]$, $\tau_3 = \operatorname{diag}[-1, -1, 1]$ form a Klein-four group and τ_1, τ_2, τ_3 are pairwise conjugate. (See Lemma 4.2 of Jha and Kallaher [7].) Hence, τ_1, τ_2, τ_3 are Baer involutions on π. Let π_1 be the Baer subplane of π fixed by τ_1. By the above lemma the plane π_1 is a translation plane with kernel a field K_1 which contains the kernel K of π.

The centralizer of τ_1 in $LC(\pi)$ contains the subgroup H of G consisting of elements whose matrix representations over $GF(w)$ have the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, w).$$

It follows that $H = SL(2, w)$. Since H centralizes τ_1, it fixes the subplane π_1. Hence, H induces a subgroup $\bar{H}$ of collineations on π_0 which is contained in the translation complement $C(\pi_0)$. By Theorem 3 of Foulser [2] the group of collineations fixing π_0 pointwise is solvable. Since $\tau_1 \in H$, it follows that $\bar{H} = H/\langle\tau_1\rangle = PSL(2, w)$. Theorem 3.1 of Jha and Kallaher [7] implies the dimension d_1 of π_1 over its kernel K_1 is divisible by 4. Since $d = 2d_1 e$, where e is the dimension of K_1 over K, by statement (ii) of Lemma 4.1, the theorem is proved.

COROLLARY 4.2.1. *Let π be a Lorimer plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. If $G = SL(n, w)$, where $n \geqslant 3$ and w is an odd prime power, is a subgroup of the linear translation complement, then 8 divides the dimension d.*

Proof. This follows directly from Theorem 4.2 since $SL(3, w) \leqslant SL(n, w)$ for all $n \geqslant 3$.

The next result, which gives a inequality relation satisfied by the Lorimer invariants and the dimension, follows from results of Harris and Hering [4] on representations of $SL(n, w)$.

THEOREM 4.3. *Let π be a Lorimer plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p a prime, and let n and w be Lorimer invariants of π. The following statements hold:*

(i) *If $w = u^s$, where $s \geqslant 1$ and u is a prime with $(u, p) = 1$, then $w^{n-1} \leqslant 2d + 1$, unless either $n = 3$, $w = 2$ or $n = 4$, $w = 2$.*

(ii) *If $w = p^s$ with $s \geqslant 1$, then $n \leqslant 2d$.*

Proof. Assume $(w, p) = 1$. By Theorem 4.3 of Harris and Hering [4], if neither $n = 3$, $w = 3$ nor $n = 4$, $w = 2$ holds, then a representation of $SL(n, w)$ on a vector space over $GF(q)$ with $(w, q) = 1$ occurs only if the vector space has dimension at least $w^n - 1$. In a Lorimer plane the vector space dimension is $2d$. Statement (i) follows. Statement (ii) follows from Theorem 4.4 of Harris and Hering [4].

COROLLARY 4.3.1. *Let π be a Lorimer plane of dimension d over its kernel $K = GF(q)$, where $q = 2^k$ with $k \geqslant 1$, and let n and w be Lorimer invariants of π. The following statements hold:*

(i) *If $w = u^s$, where $s \geqslant 1$ and u is an odd prime, then $w^{n-1} \leqslant 2d + 1$.*

(ii) *If $w = 2^s$ with $s \geqslant 1$, then $n \leqslant 2d$.*

In particular, the dimension is at least 4.

Proof. Statements (i) and (ii) follow directly from Theorem 4.3. The last sentence follows from Theorem 4.1 of Jha and Kallaher [7].

The last result of this section restricts the action of the group $SL(n, w)$ on a Lorimer plane.

THEOREM 4.4. *Let π be a Lorimer plane of dimension d over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p a prime, and let n and w be Lorimer invariants of π. Assume $p \nmid (n, w-1)$. The following statements hold:*

(i) *The group $G = SL(n, w)$ acting on π contains no affine elations.*

(ii) *If $p > 3$, the group $G = SL(n, w)$ acting on π contains no Baer p-elements.*

Proof. Recall that $n \geqslant 3$. Assume G contains non-trivial affine elations. Let E be the subgroup generated by all the affine elations in G. Then $1 \neq E \trianglelefteq G$ and $p \mid |E|$. Since $(n, w-1)$ is the order of the center of $SL(n, w)$ and $p \nmid (n, w-1)$ by assumption, it follows that $E = G = SL(n, w)$. (See pp. 177–185 of Huppert [5].) But this contradicts Theorem 35.10 of Lüneburg [12]. Hence G contains no non-trivial affine elations.

Assume $p > 3$, and assume also that G contains non-trivial Baer p-elements. If L is the subgroup generated by the Baer p-elements in G, then $1 \neq L \trianglelefteq G$ and $p \mid |L|$. It follows that $L = G = SL(n, w)$. But this contradicts Corollary 4.2 of Foulser [3]. Hence, G contains no non-trivial Baer p-elements.

Note that the assumption $p \leftarrow (n, w-1)$ holds if $w = p^s$ for some $s \geqslant 1$ or if $(n, w-1) = 1$, that is, if $SL(n, w)$ is simple.

5. LORIMER PLANES OF DIMENSION FOUR

By the results in the provious section the smallest dimension for a Lorimer plane is four. The known Lorimer planes both have dimension four over the kernel $K = GF(2)$. In this section Lorimer planes of dimension four are investigated and the possible Lorimer invariants are given. Also, Lorimer's problem is considered and it is shown that q, the order of the kernel, is 2^k for some $k \geqslant 1$ and the Lorimer invariant $n = 3$.

The first result investigates Lorimer planes of dimension four and odd characteristic.

THEOREM 5.1. *Let π be a Lorimer plane of dimension 4 over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p an odd prime. If n and w are Lorimer invariants of π, then $n = 3$ and $w = 2$.*

Proof. By Corollary 4.2.1 the invariant w must be even. By statement (i) of Theorem 4.3 one of the following occurs: (a) $w^{n-1} \leqslant 2d + 1$, (b) $n = 3$, $w = 2$, (c) $n = 4$, $w = 2$. If (a) holds, then $d = 4$ implies only (b) or (c) can hold. Case (c) cannot occur by statement (iii) of Theorem 4.1.

The second result investigates Lorimer planes of dimension four and even characteristic.

THEOREM 5.2. *Let π be a Lorimer plane of dimension 4 over its kernel $K = GF(q)$, where $q = 2^k$ with $k \geqslant 1$. If n and w are Lorimer invariants of π, then either* (i) *$n = w = 3$ or* (ii) *$n \leqslant 0$ and $w = 2^s$ with $s \geqslant 1$.*

Proof. Assume $w = u^s$ with $s \geqslant 1$ and u an odd prime. By statement (i) of Corollary 4.3.1 the inequality $w^{n-1} \leqslant 2d + 1 = 9$ holds. It follows that $n = w = 3$. If $w = 2^s$ with $s \geqslant 1$, then $n \leqslant 8$ by statement (ii) of Corollary 4.2.1.

It seems unlikely that either the conclusion of Theorem 5.1 or conclusion (i) of Theorem 5.2 can actually occur. Attempts to rule them out, however, have not succeeded. With regard to conclusion (ii) of Theorem 5.2, attempts to limit the integer s have shown so far that $s \leqslant k$. Note that the Lorimer–Rahilly and Johnson–Walker planes both satisfy conclusion (ii) of Theorem 5.2 with $s = k = 1$.

The last result of this article considers Lorimer's problem concerning translation planes of dimension 4 over their kernel $K = GF(q)$ having a collineation group $G = SL(n, q)$ with $n \geqslant 3$. The following theorem shows that for such planes $q = 2^k$ and $n = 3$.

THEOREM 5.3. *Let π be a translation plane of dimension 4 over its kernel $K = GF(q)$, where $q = p^k$ with $k \geqslant 1$ and p a prime. If π admits $G = SL(n, q)$ as a subgroup of its linear translation complement $LC(\pi)$, then $p = 2$ and $n = 3$.*

Proof. The plane π is a Lorimer plane with Lorimer invariants n and $w = q$. Thus, Theorems 5.1 and 5.2 apply. It follows that $n \leqslant 8$ and $p = 2$.

Now $SL(n, q)$, where $q = 2^k$, has as its Sylow 2-subgroups 2-groups of order q^t, where $t = \frac{1}{2}n(n-1)$. (See pp. 381–382 of Huppert [5].) Assume $n \geqslant 5$. On the line ℓ_∞ a Sylow 2-group S of G fixes a point V since ℓ_∞ has $q+1$ points. Furthermore, since $\ell_\infty - \{V\}$ has q^4 points, there is a second point U on ℓ_∞ and a subgroup S_1 of S having order q^{t-4} which fixes U. Since S_1 fixes the origin $\mathcal{O}$ and q is even, it follows that S_1 fixes pointwise a subplane of order q. (Recall that S_1 is in $LC(\pi)$ and, hence, commutes with the kernel homologies.) Thus, S_1 is a group of order q^{t-4} fixing pointwise a subplane of order q. By Proposition 6.8 of Jha [6], $q^{t-4} \leqslant q^{4-1} = q^3$ or $t \leqslant 7$. This gives a contradiction since $n \geqslant 5$ implies $t \geqslant 10$. Thus, $n = 3$ or $n = 4$.

Assume $n = 4$. Then $t = 6$. Consider the action of S on the line $\ell = \mathcal{O}V$ through the origin. The group S fixes q points of ℓ since the fixed points of S form a subspace over the kernel $K = GF(q)$, and S permutes the remaining $q^4 - q = q(q^3 - 1)$ affine points of ℓ. It follows that S has a subgroup S_1 of order q^5 fixing at least $q+1$ affine points of $\ell = \mathcal{O}V$. Hence, S_1 fixes at least q^2 affine points since its fixed points form a subspace over K. Since $q^4 - q^2 = q^2(q^2 - 1)$ the group S_1 has a subgroup S_2 of order at least q^3 fixing at least $q^2 + 1$ affine points on ℓ. Let σ be an involution in S_2. Since σ must fix q^3 affine points of ℓ and $q^3 > q^2 = \sqrt{q^4}$, the involution σ is an affine elation with axis ℓ. But this contradicts statement (i) of Theorem 4.4. Thus, $n = 3$ and this proves the theorem.

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